Given the non-homogeneous linear ODE with constant coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y^{(1)} + a_0 y = b(x)$$

where $b(x)$ consists only of sums and products of x^k, e^{mx}, $\cos nx$, $\sin nx$ and constants, where k is a non-negative integer, and m and n are constants:

eg. $y^{(7)} - y^{(5)} - 2y^{(4)} + 2y^{(3)} = x^2(1-2e^x) + xe^{-x}(3x^2e^{2x} - 4\cos x)$

1. Solve the corresponding homogeneous linear ODE $a_n y^{(n)} + a_{n-1} y^{(n-1)} + a_{n-2} y^{(n-2)} + \cdots + a_1 y^{(1)} + a_0 y = 0$ using the characteristic polynomial.

   Call the general solution y_h.

eg.

$y^{(7)} - y^{(5)} - 2y^{(4)} + 2y^{(3)} = 0$	
$r^7 - r^5 - 2r^4 + 2r^3 = 0$	
$r^3(r^4 - r^2 - 2r + 2) = 0$	Greatest common factor
$r^3(r^2(r^2-1) - 2(r-1)) = 0$	Factor by grouping
$r^3(r^2(r+1)(r-1) - 2(r-1)) = 0$	
$r^3(r-1)(r^2(r+1) - 2) = 0$	Greatest common factor
$r^3(r-1)(r^3 + r^2 - 2) = 0$	$r^3 + r^2 - 2 = 0$ when $r = 1$, so $r-1$ is a factor
$r^3(r-1)(r-1)(r^2 + 2r + 2) = 0$	Synthetic division
$r^3(r-1)^2(r^2 + 2r + 2) = 0$	
$r = 0, 0, 0, 1, 1, -1 \pm i$	Quadratic formula

$$y_h = c_0 + c_1 x + c_2 x^2 + c_3 e^x + c_4 x e^x + c_5 e^{-x}\cos x + c_6 e^{-x}\sin x$$

2. Distribute all multiplications in $b(x)$, and write $b(x)$ strictly as a sum of product terms.

eg. $b(x) = x^2 - 2x^2e^x + 3x^3e^x - 4xe^{-x}\cos x$

   Group the terms of $b(x)$ which have all the same factors, with the exception of constant factors and factors of the form x^k.

eg. $b(x) = (x^2) + (-2x^2e^x + 3x^3e^x) + (-4xe^{-x}\cos x)$

Factor out all those same factors from each group.

Each group of $b(x)$ should now be a sum of terms of the form $c_k x^k$ (ie. polynomials, possibly consisting of just one constant), multiplied (possibly) by factors not containing . x^k .

eg. $b(x) = (x^2) + (3x^3 - 2x^2)e^x + (-4x)e^{-x}\cos x$

3. For each group in $b(x)$, include that group in y_p, but

[i] replace any polynomial factors with general polynomials of the same degree using undetermined coefficients,

eg. $y_p = (Ax^2 + Bx + C) + (Dx^3 + Ex^2 + Fx + G)e^x + (Hx + J)e^{-x}\cos x$

[ii] if that group now contains any term that also appears in y_h, multiply that group by x^d, where d is the smallest positive integer power such that the new group no longer contains any like-terms in y_h.

eg. 1^{st} group:

$Ax^2 + Bx + C$	has like-terms x^2, x and a constant in y_h
$x(Ax^2 + Bx + C) = Ax^3 + Bx^2 + Cx$	has like-terms x^2 and x in y_h
$x^2(Ax^2 + Bx + C) = Ax^4 + Bx^3 + Cx^2$	has like-term x^2 in y_h
$x^3(Ax^2 + Bx + C) = Ax^5 + Bx^4 + Cx^3$	has no like-terms in y_h

2^{nd} group:

$(Dx^3 + Ex^2 + Fx + G)e^x = Dx^3e^x + Ex^2e^x + Fxe^x + Ge^x$	has like-terms xe^x and e^x in y_h
$x(Dx^3 + Ex^2 + Fx + G)e^x = Dx^4e^x + Ex^3e^x + Fx^2e^x + Gxe^x$	has like-term xe^x in y_h
$x^2(Dx^3 + Ex^2 + Fx + G)e^x = Dx^5e^x + Ex^4e^x + Fx^3e^x + Gx^2e^x$	has no like-terms in y_h

3^{rd} group:

$(Hx + J)e^{-x}\cos x = Hxe^{-x}\cos x + Je^{-x}\cos x$	has like-term $e^{-x}\cos x$ in y_h
$x(Hx + J)e^{-x}\cos x = Hx^2e^{-x}\cos x + Jxe^{-x}\cos x$	has no like-terms in y_h

$$y_p = x^3(Ax^2 + Bx + C) + x^2(Dx^3 + Ex^2 + Fx + G)e^x + x(Hx + J)e^{-x}\cos x$$

4. Differentiate y_p, ignoring all coefficients. Only the like-terms are relevant.

If all like-terms in the derivative are already in y_p or y_h, jump to step 5.

If any new like-terms appear that are not already in y_p nor y_h,
add them into y_p with new undetermined coefficients, go back to the start of step 4 and repeat.

eg. $y_h = c_0 + c_1x + c_2x^2 + c_3e^x + c_4xe^x + c_5e^{-x}\cos x + c_6e^{-x}\sin x$

$y_p = x^3(Ax^2 + Bx + C) + x^2(Dx^3 + Ex^2 + Fx + G)e^x + x(Hx + J)e^{-x}\cos x$

$y_p = (Ax^5 + Bx^4 + Cx^3) + (Dx^5 + Ex^4 + Fx^3 + Gx^2)e^x + (Hx^2 + Jx)e^{-x}\cos x$

$y_p = Ax^5 + Bx^4 + Cx^3 + Dx^5e^x + Ex^4e^x + Fx^3e^x + Gx^2e^x + Hx^2e^{-x}\cos x + Jxe^{-x}\cos x$

The derivative of y_p contains like-terms

$x^4, x^3, x^5e^x, x^4e^x, x^3e^x, x^2e^x, xe^{-x}\cos x, x^2e^{-x}\cos x$ which are already in y_p,

$x^2, xe^x, e^{-x}\cos x$ which are already in y_h,

and $x^2e^{-x}\sin x, xe^{-x}\sin x$ which are not in y_p nor y_h

Adding the new like-terms into y_p with new undetermined coefficients,

$y_p = Ax^5 + Bx^4 + Cx^3 + Dx^5e^x + Ex^4e^x + Fx^3e^x + Gx^2e^x + Hx^2e^{-x}\cos x + Jxe^{-x}\cos x + Kx^2e^{-x}\sin x + Lxe^{-x}\sin x$

The derivative of this new y_p contains all the like-terms from before (which are now all in y_p), plus

$e^{-x}\sin x$ which is already in y_h,

and no new like-terms which are not in y_p nor y_h

So, $y_p = Ax^5 + Bx^4 + Cx^3 + Dx^5e^x + Ex^4e^x + Fx^3e^x + Gx^2e^x + Hx^2e^{-x}\cos x + Jxe^{-x}\cos x + Kx^2e^{-x}\sin x + Lxe^{-x}\sin x$

$y_p = (Ax^5 + Bx^4 + Cx^3) + (Dx^5 + Ex^4 + Fx^3 + Gx^2)e^x + (Hx^2 + Jx)e^{-x}\cos x + (Kx^2 + Lx)e^{-x}\sin x$

NOTE:
Because of our original restriction on $b(x)$, all of step 4 can be significantly simplified to just the following:
if a group contains a factor $\cos nx$,
add the same group into y_p with $\sin nx$ replacing $\cos nx$ using different undetermined coefficients, unless that group is already in y_p;
similarly, if a group contains a factor $\sin nx$,
add the same group into y_p with $\cos nx$ replacing $\sin nx$ using different undetermined coefficients, unless that group is already in y_p.

eg. $y_p = (Ax^5 + Bx^4 + Cx^3) + (Dx^5 + Ex^4 + Fx^3 + Gx^2)e^x + (Hx^2 + Jx)e^{-x}\cos x$

The group $(Hx^2 + Jx)e^{-x}\cos x$ has a factor of $\cos x$, so add $(Kx^2 + Lx)e^{-x}\sin x$ into y_p since it isn't already in y_p

$y_p = (Ax^5 + Bx^4 + Cx^3) + (Dx^5 + Ex^4 + Fx^3 + Gx^2) + (Hx^2 + Jx)e^{-x}\cos x + (Kx^2 + Lx)e^{-x}\sin x$

5. Substitute y_p into the differential equation to find the values of the undetermined coefficients.
Rewrite y_p using the values of the coefficients found.

NOTE:
At the end of step 4, the format of y_p is called the **form of the particular solution**.

At the end of step 5, the resulting y_p is called a **particular solution**.

When you add a particular solution to the y_h in step 1, the resulting expression is called the **general solution**.